Introduction to Approximation theory 2023: Assignment I

1. [Schwartz class] Prove that if $\varphi \in S$ then,
   (i) $x^\alpha \varphi(x) \in S$, $\forall \alpha \in \mathbb{Z}_+^n$.
   (ii) $\partial^\alpha \varphi \in L_p(\mathbb{R}^n)$, for any $0 < p \le \infty$, $\alpha \in \mathbb{Z}_+^n$.
   (iii) $\hat{\varphi} \in \mathcal{S}$ (Hint: you may prove the case $n = 1$. Use integration by parts of $\int_{-\infty}^{\infty} \varphi(x) e^{-iwx} dx$).

2. [Distributional derivative] Prove using the definition of the distributional derivative that

$$H'(x) = \begin{cases} 1, & -1 \le x < 0, \\ -1, & 0 \le x \le 1, \\ 0, & \text{else}, \end{cases} \qquad \text{where } H(x) := \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & \text{else}. \end{cases}$$

3. Let $f, g \in L_2(\mathbb{T})$. Prove that
   (i) $f * g \in L_2(\mathbb{T})$.
   (ii) For each $k \in \mathbb{Z}$, $(f * g)^\wedge(k) = \hat{f}(k)\hat{g}(k)$.

4. Let $f(x) := \sum_{m=1}^{M} c_m \mathbf{1}_{[2m,2m+1]}(x)$. Compute the modulus $\omega_1(f,t)_p$, for all $0 < t < 1/2$, and $0 < p \le \infty$.

5. Prove the following equality for any $N \ge 1$, $x, h \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$,

$$\Delta_{Nh}^r(f,x) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \Delta_h^r(f, x + k_1 h + \ldots + k_r h).$$

Hint: recall we proved in class for $r = 1$. Now apply induction on r. Make sure the notations are correct.