Besov Spaces

Continuous definition

Let $\alpha > 0$ and $0 < q \le \infty$. Let $r \ge \lfloor \alpha \rfloor + 1$. The Besov space $B_q^\alpha \left(L_p \left(\Omega \right) \right)$ is the collection of functions $f \in L_p \left(\Omega \right)$ for which

$$|f|_{B_q^\alpha(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty \left[t^{-\alpha} \omega_r \left(f, t \right)_p \right]^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{-\alpha} \omega_r \left(f, t \right)_p, & q = \infty. \end{cases}$$

is finite. The norm is

$$\|f\|_{B_q^\alpha(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + |f|_{B_q^\alpha(L_p(\Omega))}. \tag{1.1}$$

Theorem The space does not depend on $r \ge \lfloor \alpha \rfloor + 1$.

Theorem For a bounded domain we can equivalently integrate the semi norm over the limits $[0,1]$.

Proof If Ω is compact, then we have $\omega_r \left(f, t \right)_p \equiv const$ for $t \ge diam(\Omega)$. Therefore

$$\omega_r \left(f, 1/2 \right)_p \le \omega_r \left(f, t \right)_p \le C \omega_r \left(f, 1/2 \right)_p, \ 1/2 \le t \le \infty.$$

Thus,

$$\begin{aligned} \int_1^\infty \left[t^{-\alpha} \omega_r \left(f, t \right)_p \right]^q \frac{dt}{t} &\le C^q \left(\omega_r \left(f, 1/2 \right)_p \right)^q \int_1^\infty t^{-q\alpha - 1} dt \\ &\le C \left(\alpha, q, \Omega \right) \int_{1/2}^1 \left[t^{-\alpha} \omega_r \left(f, t \right)_p \right]^q \frac{dt}{t}. \end{aligned}$$

♦

Lemma For any domain, integrating the semi norm over the limits $[0,1]$ gives a quasi-norm equivalent to (1.1)

Proof We can bound the integration of the semi-norm over $[1,\infty]$ by the p-norm

$$\begin{aligned} \int_1^\infty \left[t^{-\alpha} \omega_r \left(f, t \right)_p \right]^q \frac{dt}{t} &\le \left(2^r \|f\|_p \right)^q \int_1^\infty t^{-q\alpha - 1} dt \\ &= C \left(\alpha, q \right) \|f\|_p^q. \end{aligned}$$

♦

Theorem $B_{q_1}^{\alpha_1} \left(L_p \right) \subseteq B_{q_2}^{\alpha_2} \left(L_p \right)$ if $\alpha_2 < \alpha_1$.

Proof $(q_1 = q_2)$ For $0 < t \le 1$, $t^{-\alpha_2} \le t^{-\alpha_1}$. We also note that we may use $r_1 = \lfloor \alpha_1 \rfloor + 1 \ge \lfloor \alpha_2 \rfloor + 1 = r_2$ to equivalently define $B_{q_2}^{\alpha_2}(L_p)$

♦

Theorem $W_p^m \subseteq B_q^\alpha(L_p)$, $\forall \alpha < m$, $1 \le p \le \infty$, $0 < q \le \infty$.

Proof Let $g \in W_p^m(\Omega)$. Then $r := \lfloor \alpha \rfloor + 1 \le m$. For any $r \le m$, $g \in W_p^r(\Omega)$, and so for any $t > 0$,

$$\omega_r(g,t)_p \le t^r |g|_{r,p}.$$

It is sufficient to integrate the semi-norm only over $[0,1]$. Since $r > \alpha$,

$$\begin{aligned}\int_0^1 \left[t^{-\alpha}\omega_r(g,t)_p\right]^q \frac{dt}{t} &\le c\int_0^1 \left[t^{-\alpha}t^r |g|_{r,p}\right]^q \frac{dt}{t} \\ &\le c|g|_{r,p}^q \int_0^1 t^{(r-\alpha)q-1}dt \\ &\le c|g|_{r,p}^q .\end{aligned}$$

♦

Discrete forms of the Besov semi-norm

Theorem Define $\varphi(t) := t^{-\alpha}\omega_r(f,t)_p$. Then for $t \in [2^{-k-1}, 2^{-k}]$, $k \in \mathbb{Z}$, we have

$$2^{-r}\varphi(2^{-k}) \le \varphi(t) \le 2^\alpha \varphi(2^{-k}).$$

Proof We use the following properties:

(i) $\omega_r(f,t)_p$ is non-decreasing

(ii) $\omega_r(f,\lambda t)_p \le (\lambda+1)^r \omega_r(f,t)_p$

The left-hand side

$$\begin{aligned}2^{-r}\varphi(2^{-k}) &= 2^{k\alpha-r}\omega_r(f,2^{-k})_p = 2^{k\alpha-r}\omega_r(f,22^{-k-1})_p \\ &\underset{\text{(ii)}}{\le} 2^{k\alpha-r}2^r\omega_r(f,2^{-k-1})_p \underset{\text{(i)}}{\le} 2^{k\alpha}\omega_r(f,t)_p \le t^{-\alpha}\omega_r(f,t)_p\end{aligned}$$

The right-hand side

$$t^{-\alpha}\omega_r(f,t)_p \underset{\text{(i)}}{\le} t^{-\alpha}\omega_r(f,2^{-k})_p \le 2^{(k+1)\alpha}\omega_r(f,2^{-k})_p \le 2^\alpha\varphi(2^{-k})$$

♦

This gives us for $k \in \mathbb{Z}$

$$\int_{2^{-k-1}}^{2^{-k}} \varphi(t)^q \frac{dt}{t} \sim \varphi(2^{-k})^q \int_{2^{-k-1}}^{2^{-k}} \frac{dt}{t} \sim \varphi(2^{-k})^q \Rightarrow \int_{2^{-k-1}}^{2^{-k}} \left(t^{-\alpha}\omega_r(f,t)_p\right)^q \frac{dt}{t} \sim \left[2^{k\alpha}\omega_r(f,2^{-k})_p\right]^q.$$

And so

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \begin{cases} \left(\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\omega_r(f,2^{-k})_p\right]^q\right)^{1/q}, & 0 < q < \infty. \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha}\omega_r(f,2^{-k})_p, & q = \infty. \end{cases}$$

$B_\tau^\alpha := B_\tau^\alpha(L_\tau)$ - We will focus on the choice $1/\tau := \alpha + 1/p$, when we are approximating functions in L_p .

Theorem $\Omega = \mathbb{R}$. Let $D := \{D_k : k \in \mathbb{Z}\}$, $D_k := \{Q = 2^{-k}[j, j+1] : j \in \mathbb{Z}\}$. Observe that $Q \in D_k \Rightarrow |Q| = 2^{-k}$. We have the equivalence

$$|f|_{B_\tau^\alpha} \sim \left(\sum_k \left(2^{k\alpha} \omega_r \left(f, 2^{-k} \right)_\tau \right)^\tau \right)^{1/\tau} \sim \left(\sum_{Q \in D} \left(|Q|^{-\alpha} \omega_r \left(f, Q \right)_\tau \right)^\tau \right)^{1/\tau} .$$

Leveraging on this equivalnce, we will generalize Besov spaces to high-dimensions and anisotropic partitions of trees over $[0,1]^n$ (replacing dyadic cubes!). Let $\mathcal{T}$ be a decision tree/binary space partition tree. We define

$$|f|_{B_\tau^\alpha(\mathcal{T})} := \left(\sum_{\Omega \in \mathcal{T}} \left(|\Omega|^{-\alpha} \omega_r \left(f, \Omega \right)_\tau \right)^\tau \right)^{1/\tau},$$

where

$$\omega_r \left(f, \Omega \right)_\tau := \sup_{h \in \mathbb{R}^n} \left\| \Delta_h^r f \left(\cdot, \Omega \right) \right\|_{L_\tau(\Omega)} .$$