What is approximation theory?

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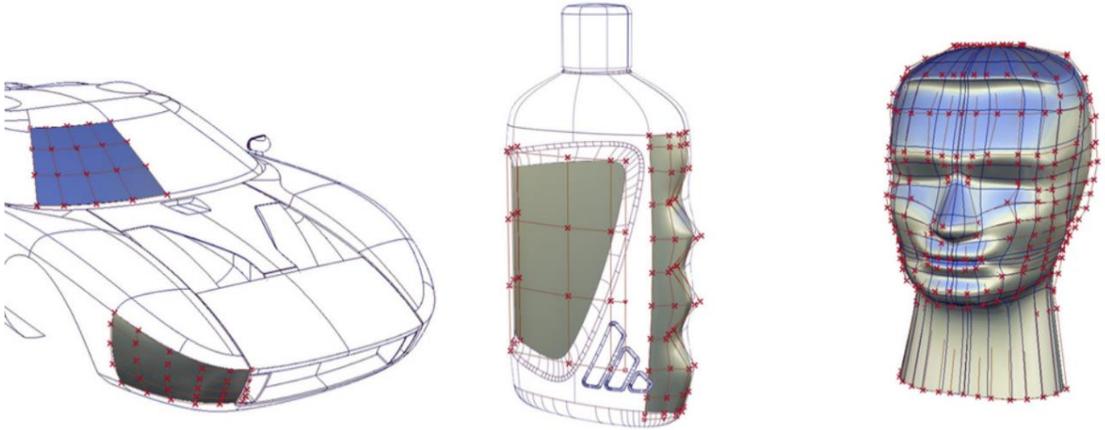
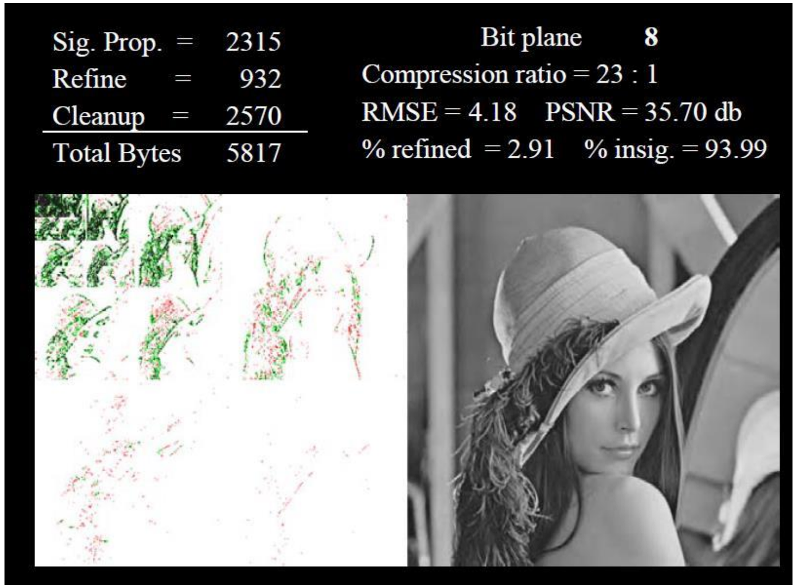
“Although this may seem a paradox, all exact science is dominated by the idea of approximation”

Bertrand Russel

“In mathematics, approximation theory is concerned with how functions can best be approximated with simpler functions, and with quantitatively characterizing the errors introduced thereby”

Wikipedia

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Refine = 932
Cleanup = 2570
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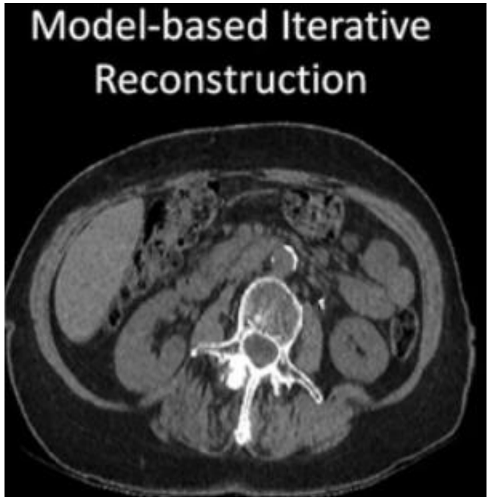
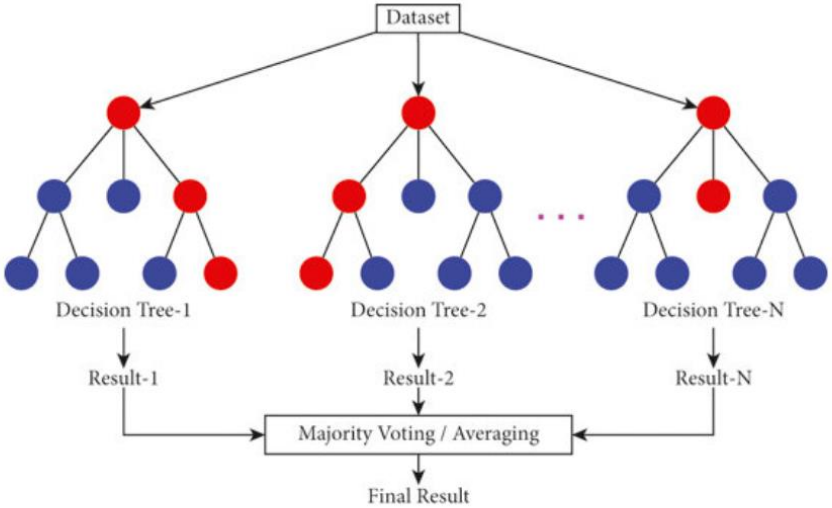
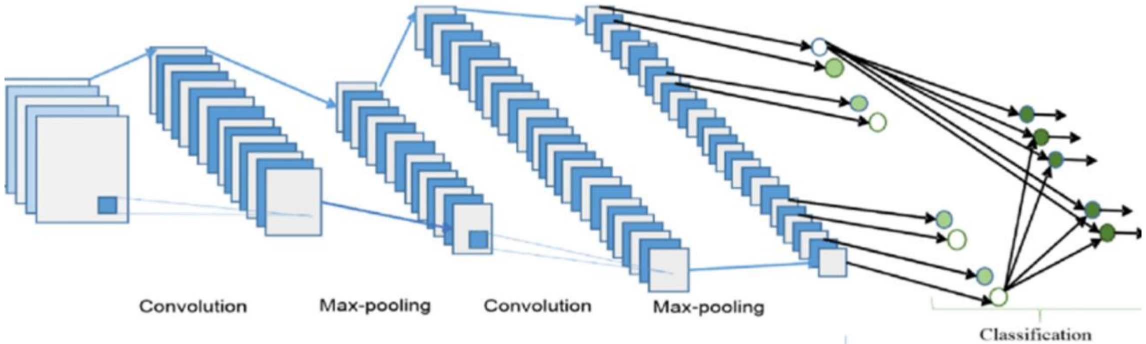
Bit plane 8
Compression ratio = 23 : 1
RMSE = 4.18 PSNR = 35.70 db
% refined = 2.91 % insig. = 93.99


Model-based Iterative Reconstruction

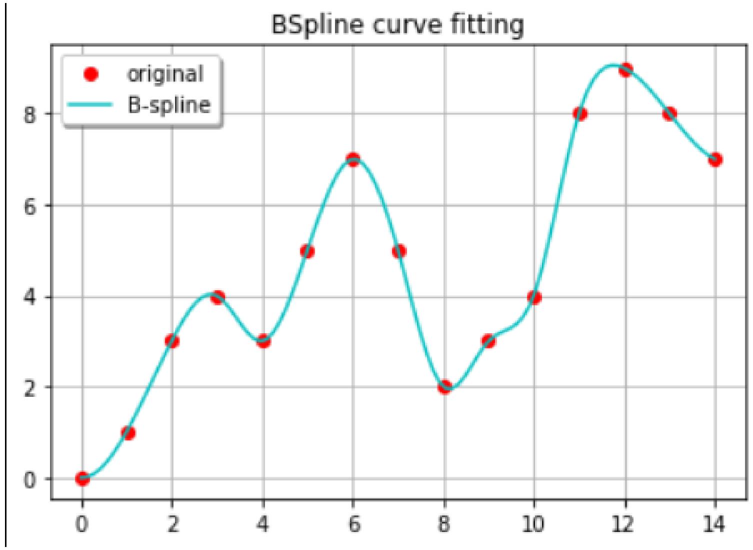

Dataset
Decision Tree-1
Decision Tree-2
Decision Tree-N
Result-1
Result-2
Result-N
Majority Voting / Averaging
Final Result

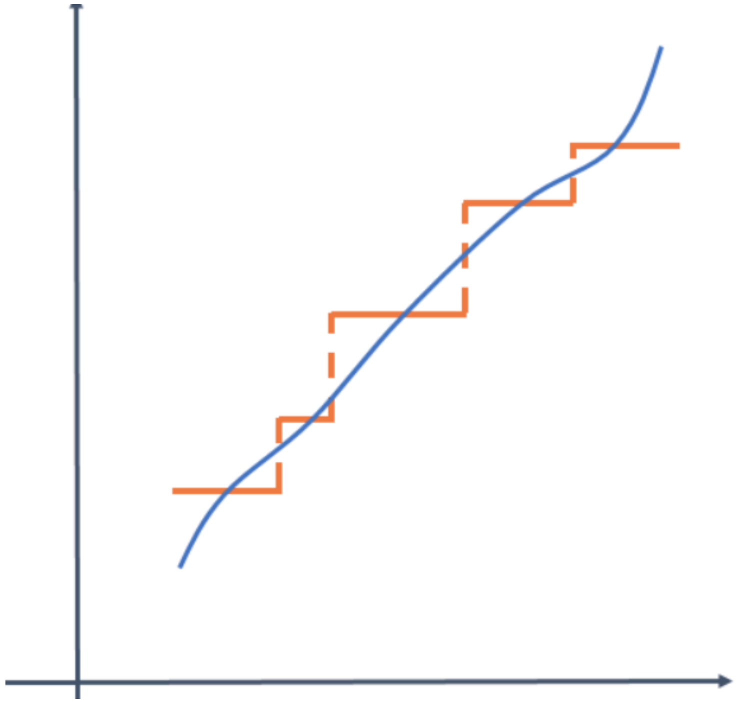

Convolution
Max-pooling
Convolution
Max-pooling
Classification

Approximation algorithms using $\sim N$ degrees of freedom

<u>Linear</u>

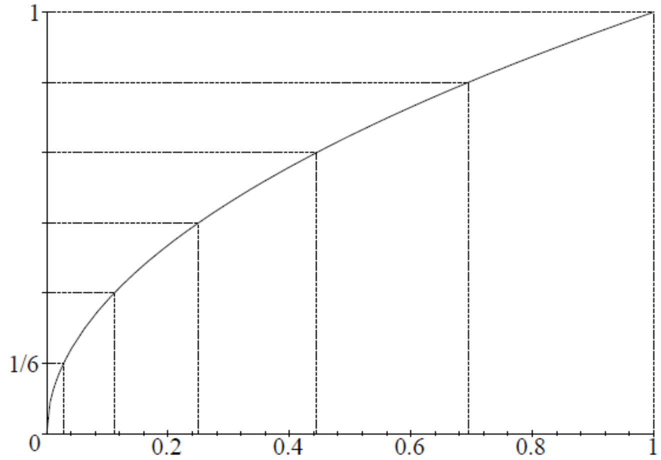
- Trigonometric polynomials of degree $\leq N$, $X = L_p[-\pi, \pi]$.
- Algebraic polynomials of degree $\leq N$, $X = L_p[-1,1]$.
- Piecewise polynomials over N uniform intervals with fixed polynomial degree, $X = L_p[0,1]$.
- Shift invariant spaces (e.g. uniform B-splines)





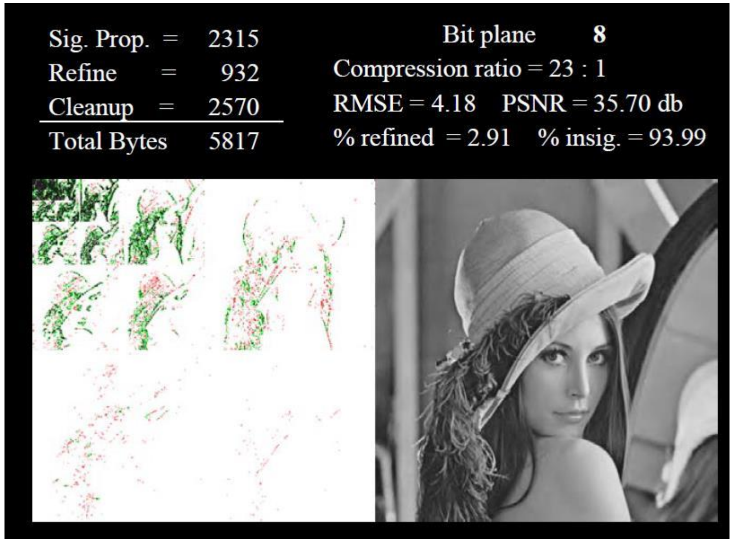
Nonlinear/Adaptive

- Rational functions of degree $\le N$, $X = L_p[-1,1]$.
- Free knot piecewise polynomials of fixed degree over N non-uniform intervals, $X = L_p[0,1]$.



Approximation of $\sqrt{x}$ using free knots

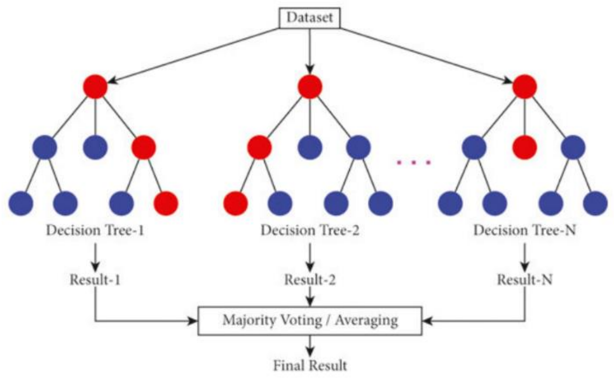
- N-term wavelets $\Sigma_N = \left\{ \sum_{\#I \le N} c_I \psi_I \right\}$, $X = L_2$.



Machine learning examples

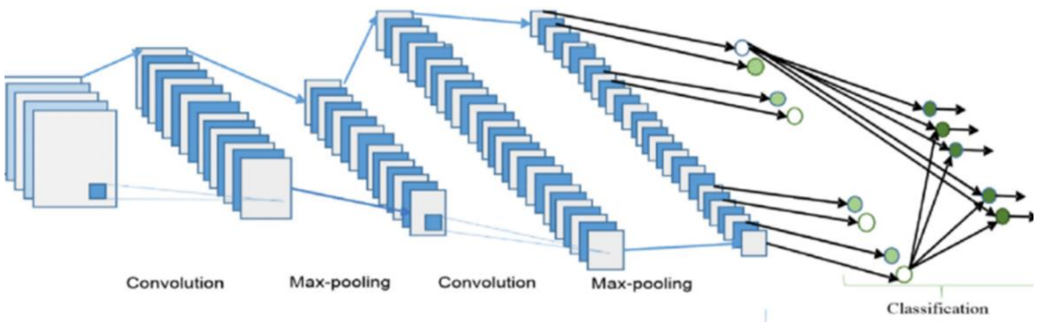
High dimensional generalizations of free knot piecewise polynomials:

- Decision trees
- Random Forest
- Gradient Boosting



Neural networks

- Researched by the approximation theoretical community since the 50s(!)
- A new wave of AI is creating many open problems!



Approximation spaces

Let Φ_N be the collection of functions we use for approximation with $\sim N$ degree of freedom. Define

$$E_N(f) := \inf_{g \in \Phi_N} \|f - g\|_X$$

For $\alpha > 0$, $0 < q \le \infty$, $f \in X$,

$$|f|_{A_q^\alpha} := \begin{cases} \left(\sum_{N=1}^{\infty} \left[N^\alpha E_N(f) \right]^q \frac{1}{N} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{N \ge 1} N^\alpha E_N(f), & q = \infty. \end{cases}$$

$$\|f\|_{A_q^\alpha} := \|f\|_X + |f|_{A_q^\alpha}.$$

Goal: Fully characterize approximation spaces by function (smoothness) spaces ($\Leftrightarrow$)

Characterization Example – Trigonometric polynomials

- Besov spaces are smoothness spaces that contain in some cases piecewise smooth functions that are not differentiable(!)
- They contain other simpler smoothness spaces as special cases (Lipschitz, Sobolev)
- For trigonometric polynomial approximation we have the characterization

$$A_q^\alpha\left(L_p\right) \sim B_q^\alpha\left(L_p\right).$$

- This means that if $f \in B_q^\alpha\left(L_p\right)$, then $f \in A_q^\alpha\left(L_p\right)$ and vice versa
- Technically we have something stronger. There exists an absolute constant $c_1 > 0$, such that $\|f\|_{A_q^\alpha(L_p)} \le c_1 \|f\|_{B_q^\alpha(L_p)}$ and vice versa.