Approximation Theory 2023 – List of theorems for the exam

[30%] Theorem [Hölder] $1\le p\le\infty$, $f\in L_p, g\in L_{p'}$

$$\left|\int_\Omega fg\right|\le\int_\Omega |fg|=\|fg\|_1\le\|f\|_p\|g\|_{p'} \qquad \frac{1}{p}+\frac{1}{p'}=1.$$

Lemma Young's inequality for products,

$$ab\le\frac{a^p}{p}+\frac{b^{p'}}{p'}, \qquad \frac{1}{p}+\frac{1}{p'}=1,\ \forall a,b\ge 0.$$

Proof of lemma The logarithmic function is concave. Therefore

$$\begin{aligned}\log\left(\frac{1}{p}a^p+\frac{1}{p'}b^{p'}\right)&=\log\left(\frac{1}{p}a^p+\left(1-\frac{1}{p}\right)b^{p'}\right)\\&\ge\frac{1}{p}\log\left(a^p\right)+\frac{1}{p'}\log\left(b^{p'}\right)\\&=\log(a)+\log(b)=\log(ab).\end{aligned}$$

Since the logarithmic function is increasing, we are done (or we take exp on both sides).

□

Proof of Hölder's theorem If $p=\infty$

$$\int_\Omega |fg|\le\|f\|_\infty\int_\Omega|g|\le\|f\|_\infty\|g\|_1.$$

The proof is similar for $p=1$. So, assume now $1<p<\infty$ and $\|f\|_p=\|g\|_{p'}=1$.
Integrating pointwise and applying Young's inequality almost everywhere, gives

$$\begin{aligned}\int_\Omega|f(x)g(x)|dx&\le\int_\Omega\left(\frac{|f(x)|^p}{p}+\frac{|g(x)|^{p'}}{p'}\right)dx\\&=\frac{1}{p}\int_\Omega|f(x)|^p\,dx+\frac{1}{p'}\int_\Omega|g(x)|^{p'}\,dx\\&=\frac{1}{p}+\frac{1}{p'}=1\end{aligned}$$

Now assuming $f,g\ne 0$ (else, we are done)

$$\int_\Omega\frac{|f(x)|}{\|f\|_p}\frac{|g(x)|}{\|g\|_{p'}}dx\le 1\Rightarrow\int_\Omega|fg|\le\|f\|_p\|g\|_{p'}$$

□

[20%] Theorem If $|\Omega| < \infty$, $0 < q < p$, $f \in L_p(\Omega)$ then

$$\| f\|_{L_q(\Omega)} \le |\Omega|^{1/q-1/p} \| f\|_{L_p(\Omega)}.$$

Proof For $p = \infty$

$$\| f\|_q = \left(\int_\Omega |f(x)|^q dx\right)^{1/q} \le \| f\|_\infty \left(\int_\Omega dx\right)^{1/q} = \| f\|_\infty |\Omega|^{1/q}.$$

For $q < p < \infty$ define $r := p/q \ge 1$

$$\| f\|_q^q = \int_\Omega |f|^q = \int_\Omega |f|^q 1 \underset{Holder}{\le} \left(\int_\Omega \left(|f|^q\right)^r\right)^{1/r} \left(\int_\Omega 1^{r'}\right)^{1/r'}$$

$$= \left(\int_\Omega |f|^p\right)^{q/p} |\Omega|^{1-q/p}$$

□

[30%] Theorem For $0 < p < 1$, we have

(i) $\left\|\sum_k f_k\right\|_p^p \le \sum_k \| f_k\|_p^p$.

(ii) $\| f+g\|_p \le 2^{1/p-1}\left(\| f\|_p + \|g\|_p\right)$ or in general $\left\|\sum_{k=1}^N f_k\right\|_p \le N^{1/p-1}\sum_{j=1}^N \| f_k\|_p$.

Proof We first show that the quasi-triangle inequality (ii) is derived from (i). Observe first

$$\left.\begin{aligned} 1 < r := \frac{1}{p} < \infty \\ \frac{1}{r} + \frac{1}{r'} = 1 \end{aligned}\right\} \Rightarrow r' = \frac{1}{1-p}.$$

Then

$$\left\|\sum_{k=1}^N f_k\right\|_p \underset{(i)}{\le} \left(\sum_{k=1}^N \| f_k\|_p^p\right)^{1/p} = \left(\sum_{k=1}^N 1\cdot \| f_k\|_p^p\right)^{1/p} \underset{\substack{\text{Discrete}\\ \text{Holder}}}{\le} \left(\sum_{k=1}^N 1^{\frac{1}{1-p}}\right)^{(1-p)1/p} \left(\sum_{k=1}^N \| f_k\|_p\right) = N^{1/p-1}\sum_{k=1}^N \| f_k\|_p .$$

To prove (i), we need the following lemma

Lemma I For $0 < p \le 1$ and any sequence of non-negative $a = \{a_k\}$,

$$\left(\sum_k a_k\right)^p \le \sum_k a_k^p$$

Proof Observe that it is sufficient to prove $(a_1 + a_2)^p \le a_1^p + a_2^p$ and then apply induction.

To prove the inequality use $h(t) := t^p + 1 - (t+1)^p$ for $t \ge 0$. $h(0) = 0$ and $h'(t) = pt^{p-1} - p(t+1)^{p-1} \ge 0$.

Therefore, $h(t) \ge 0$, for $t \ge 0$. This gives $t^p + 1 \ge (t+1)^p$. Setting $t = a_1/a_2$ gives

$$\left(\frac{a_1}{a_2}\right)^p + 1 \ge \left(\frac{a_1}{a_2}+1\right)^p \Rightarrow a_1^p + a_2^p \ge (a_1 + a_2)^p.$$

□

Proof of Theorem (i) : Simply apply the lemma pointwise for $x \in \Omega$ and then Tonelli's theorem for the exchange of integration and sum

$$\left\|\sum_k f_k\right\|_p^p \le \int_\Omega \left(\sum_k |f_k(x)|\right)^p dx \le \int_\Omega \left(\sum_k |f_k(x)|^p\right) dx = \sum_k \int_\Omega |f_k(x)|^p \, dx = \sum_k \|f_k\|_p^p .$$

□

[30%] Theorem [Fourier series approximation] Let $f \in W_2^r(\mathbb{T})$ then

$$E_N(f)_2 = \|f - S_N f\|_2 \le N^{-r} |f|_{r,2} .$$

Proof

1. Decay of the Fourier coefficients - By Parseval, for any $g \in L_2(\mathbb{T})$

$$\|g\|_{L_2(\mathbb{T})}^2 = \sum_{k=-\infty}^{\infty} |\hat{g}(k)|^2 .$$

we have

$$E_N(f)_2 = \|f - S_N f\|_{L_2(\mathbb{T})} = \sqrt{\sum_{|k| \ge N+1} \left|\hat{f}(k)\right|^2}$$

Assume first that $f \in C^r(\mathbb{T})$. We will show $\left|\hat{f}(k)\right| = |k|^{-r} \left|\left(f^{(r)}\right)^\wedge(k)\right|$. Using the continuity of f as a periodic function, integration by parts yields,

$$\begin{aligned} \hat{f}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \\ &= \frac{1}{2\pi} \left(\underbrace{\left. \frac{f(x) e^{-ikx}}{-ik} \right|_{-\pi}^{\pi}}_{=0} + \frac{1}{ik} \int_{-\pi}^{\pi} f'(x) e^{-ikx} dx \right) \\ &= \frac{1}{ik} (f')^\wedge(k). \end{aligned}$$

By repeated application of the above

$$\left|\hat{f}(k)\right| = |k|^{-r} \left|\left(f^{(r)}\right)^\wedge(k)\right|.$$

2. The estimate of the tail

$$\begin{aligned} \|f - S_N f\|_2^2 &= \sum_{|k| \ge N+1} \left|\hat{f}(k)\right|^2 \\ &\le N^{-2r} \sum_{|k| \ge N+1} |k|^{2r} \left|\hat{f}(k)\right|^2 \\ &= N^{-2r} \sum_{|k| \ge N+1} \left|\left(f^{(r)}\right)^\wedge(k)\right|^2 \\ &\le N^{-2r} \left\|f^{(r)}\right\|_2^2 . \end{aligned}$$

$$\Rightarrow E_N(f)_2 = \|f - S_N f\|_2 \le N^{-r}\|f^r\|_2 .$$

For the general case $f \in W_2^r(\mathbb{T})$ we apply a **density** argument. Let $\{f_j\}_{j=1}^{\infty}$, $f_j \in C^r(\mathbb{T})$, such that

$$\|f - f_j\|_{W_2^r} \underset{j\to\infty}{\to} 0 .$$

This implies

$$\|f - f_j\|_2 \underset{j\to\infty}{\to} 0, \quad \|f^{(r)} - f_j^{(r)}\|_2 \underset{j\to\infty}{\to} 0 .$$

Therefore

$$\begin{aligned}\|f - S_N(f)\|_2 &\le \|f - f_j\|_2 + \|f_j - S_N(f_j)\|_2 + \|S_N(f_j) - S_N(f)\|_2 \\ &\le N^{-r}\|f_j^{(r)}\|_2 + \|f - f_j\|_2 + \|S_N(f - f_j)\|_2 \\ &\le N^{-r}\|f_j^{(r)}\|_2 + 2\|f - f_j\|_2 \underset{j\to\infty}{\to} N^{-r}\|f^{(r)}\|_2\end{aligned}$$

□

Def A ***summability kernel*** is a sequence $\{h_N\}$ satisfying:

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} h_N(x)\,dx = 1$

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |h_N(x)|\,dx \le C$.

(ii) For all $0 < \delta < \pi$, $\lim_{N\to\infty}\int_{|x|\ge\delta} |h_N(x)|\,dx = 0$

[30%] Theorem For a summability kernel $\{h_N\}$ and $f \in C(\mathbb{T})$, we have

$$\|f - h_N * f\|_{C(\mathbb{T})} = \max_{-\pi\le x\le\pi} |f(x) - h_N * f(x)| \underset{N\to\infty}{\to} 0 .$$

Proof Assume $x = 0$. Let $\varepsilon > 0$. From the uniform continuity of f, there exists $0 < \delta < \pi$, such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

$$\begin{aligned}h_N * f(0) - f(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h_N(t)\big(f(-t) - f(0)\big)dt \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta} h_N(t)\big(f(-t) - f(0)\big)dt + \frac{1}{2\pi}\int_{|x|\ge\delta} h_N(t)\big(f(-t) - f(0)\big)dt\end{aligned}$$

Now

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-\delta}^{\delta} h_N(t)\big(f(-t) - f(0)\big)dt\right| &\le \max_{-\delta\le y\le\delta}|f(y) - f(0)|\frac{1}{2\pi}\int_{-\pi}^{\pi}|h_N(t)|\,dt \\ &\le C\varepsilon.\end{aligned}$$

Therefore

$$|h_N * f(0) - f(0)| \le C\varepsilon + 2\|f\|_\infty \frac{1}{2\pi}\int_{|x|\ge\delta}|h_N(t)|\,dt \underset{N\to\infty}{\to} C\varepsilon .$$

For $x \neq 0$, define $\tilde{f}(t) = f(t+x)$. Then

$$h_N * \tilde{f}(0) = \frac{1}{2\pi}\int \tilde{f}(0-y)h_N(y)dy = \frac{1}{2\pi}\int f(0-y+x)h_N(y)dy$$
$$= \frac{1}{2\pi}\int f(x-y)h_N(y)dy = h_N * f(x).$$

We now apply the first part of the proof for $\tilde{f}$ at 0, observing that $\|\tilde{f}\|_\infty = \|f\|_\infty$ and that for any $\varepsilon > 0$, we can use the same $\delta > 0$ we used for f. Hence, the approximation and convergence are in fact uniform for all $x \in \mathbb{T}$

□

[30%] Theorem For $f \in W_p^1(\mathbb{R})$, $1 \le p \le \infty$,

$$E\left(f, S(N_1)^h\right)_{L_p(\mathbb{R})} := \inf_{g \in S(N_1)^h} \|f-g\|_{L_p(\mathbb{R})} \le h|f|_{W_p^1(\mathbb{R})}.$$

Proof First assume $f \in C^1(\mathbb{R}) \cap W_p^1(\mathbb{R})$. Let's take the interval $[kh,(k+1)h]$. Then, for $p = \infty$

$$|f(x) - f(kh)| = \left|\int_{kh}^{x} f'(u)du\right| \le h \sup_{kh \le u \le (k+1)h} |f'(u)|.$$

If we select $c_k := f(kh)$, we get the theorem for $p = \infty$ by using

$$g(x) = \sum_{k \in \mathbb{Z}} f(kh) N_1\left(h^{-1}x - k\right).$$

For $1 \le p < \infty$ we do something similar

$$|f(x) - f(kh)|^p \le \left(\int_{kh}^{(k+1)h} |f'(u)|du\right)^p, \qquad x \in [kh,(k+1)h].$$

Then

$$\int_{kh}^{(k+1)h} |f(x) - f(kh)|^p dx \le h\left(\int_{kh}^{(k+1)h} |f'(u)|du\right)^p$$
$$\le h\left(\|f'\|_{L_p([kh,(k+1)h])} \|1\|_{L_{p'}([kh,(k+1)h])}\right)^p$$
$$= hh^{p/p'} \|f'\|_{L_p([kh,(k+1)h])}^p$$
$$= h^p \|f'\|_{L_p([kh,(k+1)h])}^p.$$

$$1 + \frac{p}{p'} = 1 + p\left(1 - \frac{1}{p}\right)$$
$$= 1 + p - 1 = p$$

Therefore, with $g(x) := \sum_k f(kh) N_1\left(h^{-1}x - k\right)$, we get

$$\|f-g\|_p^p = \int_{-\infty}^{\infty} |f(x) - g(x)|^p dx = \sum_k \int_{kh}^{(k+1)h} |f(x) - f(kh)|^p dx \le \sum_k h^p \|f'\|_{L_p([kh,(k+1)h])}^p = h^p \|f'\|_p^p.$$

Now assume $f \in W_p^1(\mathbb{R})$, $1 \le p < \infty$. There exist sequences $\{f_k\}$, $f_k \in C^1(\mathbb{R}) \cap W_p^1(\mathbb{R})$, $\{g_k\}$, $g_k \in S(N_1)^h$, such that $\|f - f_k\|_{W_p^r(\mathbb{R})} \underset{k\to\infty}{\to} 0$ and $\|f_k - g_k\|_{L_p(\mathbb{R})} \le h|f_k|_{W_p^1(\mathbb{R})}$. This gives

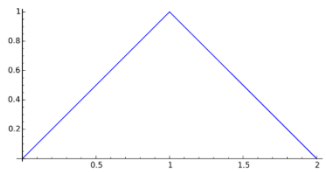
$$\begin{aligned}\|f-g_k\|_p &\le \|f-f_k\|_p + \|f_k-g_k\|_p \\ &\le \|f-f_k\|_p + h|f_k|_{1,p} \underset{k\to\infty}{\to} h|f|_{1,p}\end{aligned}$$

□

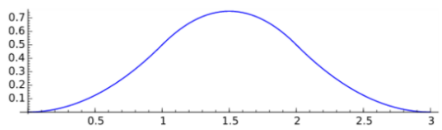
[40%] Theorem For $g \in W_p^r(\mathbb{R})$, $1 \le p \le \infty$, we have that

$$\omega_r(g,t)_{L_p(\mathbb{R})} \le Ct^r |g|_{W_p^r(\mathbb{R})}, \qquad \forall t > 0.$$

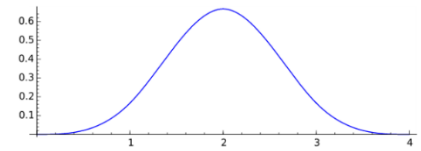
Proof Recall the B-Splines, $N_1 = \mathbf{1}_{[0,1]^n}$. In general, $N_r := N_{r-1} * N_1 = \int_{\mathbb{R}^n} N_{r-1}(x-t)N_1(t)\,dt$.



N_2



N_3



N_4

- Properties:
  - Order r
  - Support $[0,r]$
  - Piecewise polynomial of degree $r-1$ with breakpoints (knots) at the integers
  - Smoothness $r-2$, thus in Sobolev W_p^{r-1}.
  - $\int_{\mathbb{R}} N_r(x)\,dx = 1$

Here, we use the fact that for $h>0$, $\Delta_{-h}^r(f,x) = \Delta_h^r(f,x-rh)$. So W.L.G, for any $t>0$, we can work with $0<h\le t$. Define $N_r(x,h) := h^{-1}N_r(h^{-1}x)$, $h>0$. Let $g \in C^1(\mathbb{R})$. Then

$$\begin{aligned}h^{-1}\Delta_h(g,x) &= h^{-1}\left(g(x+h)-g(x)\right) \\ &= h^{-1}\int_x^{x+h} g'(u)\,du \\ &= \int_{\mathbb{R}} g'(x+u)N_1(u,h)\,du\end{aligned}$$

We claim that generally for $g \in C^r(\mathbb{R})$

$$h^{-r}\Delta_h^r(g,x) = \int_{\mathbb{R}} g^{(r)}(x+u)N_r(u,h)\,du$$

To see that we apply induction

$$
\begin{aligned}
h^{-r}\Delta_h^r(g,x) &= h^{-1}h^{-(r-1)}\left(\Delta_h^{r-1}(g,x+h)-\Delta_h^{r-1}(g,x)\right)\\
&= h^{-1}\left(\int_{\mathbb{R}} g^{(r-1)}(x+h+u)N_{r-1}(u,h)\,du-\int_{\mathbb{R}} g^{(r-1)}(x+u)N_{r-1}(u,h)\,du\right)\\
&= h^{-1}\int_x^{x+h}\int_{-\infty}^{\infty} g^{(r)}(v+u)N_{r-1}(u,h)\,du\,dv\\
&= \int_{-\infty}^{\infty} N_{r-1}(u,h)\left(h^{-1}\int_x^{x+h} g^{(r)}(v+u)\,dv\right)du\\
&= \int_{-\infty}^{\infty} N_{r-1}(u,h)\left(\int_{-\infty}^{\infty} g^{(r)}(v+u)N_1(v-x,h)\,dv\right)du\\
&\overset{v+u=x+y}{=} \int_{-\infty}^{\infty} N_{r-1}(u,h)\left(\int_{-\infty}^{\infty} g^{(r)}(x+y)N_1(y-u,h)\,dy\right)du\\
&= \int_{-\infty}^{\infty} g^{(r)}(x+y)\left(\int_{-\infty}^{\infty} N_{r-1}(u,h)N_1(y-u,h)\,du\right)dy\\
&= \int_{-\infty}^{\infty} g^{(r)}(x+y)\mathrm{N}_r(y,h)\,\mathrm{dy}
\end{aligned}
$$

First we prove for $p=\infty$

$$
\begin{aligned}
\left|\Delta_h^r(g,x)\right| &\le h^r\int_{\mathbb{R}}\left|g^{(r)}(x+u)\right|N_r(u,h)\,du\\
&\le t^r\left\|g^{(r)}\right\|_\infty \underbrace{\int_{\mathbb{R}} N_r(u,h)\,du}_{=1}\\
&= t^r\,|g|_{r,\infty}.
\end{aligned}
$$

Now, let's see the proof for $p=1$. Let $0<h\le t$

$$
\begin{aligned}
\int_{\mathbb{R}}\left|\Delta_h^r(g,x)\right|dx &\le h^r\int_{\mathbb{R}}\int_{\mathbb{R}}\left|g^{(r)}(x+u)\right|\left|N_r(u,h)\right|du\,dx\\
&\le t^r\underbrace{\int_{\mathbb{R}}\left|N_r(u,h)\right|du}_{=1}\int_{\mathbb{R}}\left|g^{(r)}(x)\right|dx\\
&\le t^r\,|g|_{W_1^r(\mathbb{R})}.
\end{aligned}
$$

For general $1\le p<\infty$ we need Minkowski's inequality. It says that for measurable non-negative functions φ,ρ

$$
\left\{\int_A\left(\int_B\varphi(y)\rho(x,y)\,dy\right)^p dx\right\}^{1/p} \le \int_B\varphi(y)\left(\int_A\rho(x,y)^p\,dx\right)^{1/p} dy
$$

Or written differently

$$
\left\|\int_B\varphi(y)\rho(\cdot,y)\,dy\right\|_{L_p(A)} \le \int_B\varphi(y)\left\|\rho(\cdot,y)\right\|_{L_p(A)} dy
$$

Using it we have

$$\int_{\mathbb{R}}\left|\Delta_h^r(g,x)\right|^p dx \le h^{pr}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|g^{(r)}(x+u)\right|\left|N_r(u,h)\right|du\right)^p dx$$
$$\le h^{pr}\left(\int_{\mathbb{R}}\left|N_r(u,h)\right|\left\|g^{(r)}(\cdot+u)\right\|_{L_p(\mathbb{R})}du\right)^p$$
$$= h^{pr}\left\|g^{(r)}\right\|_{L_p(\mathbb{R})}\left(\underbrace{\int_{\mathbb{R}}\left|N_r(u,h)\right|du}_{=1}\right)^p$$
$$\le t^{pr}\left|g\right|^p_{W_p^r(\mathbb{R})}.$$

For a general function $g\in W_p^r(\mathbb{R})$ we use the density of $C^r(\mathbb{R})\cap W_p^r(\mathbb{R})$ in $W_p^r(\mathbb{R})$. Let $\{g_j\}_{j=1}^{\infty}$, $g_j\in C^r(\mathbb{R})\cap W_p^r(\mathbb{R})$, $\left\|g-g_j\right\|_{W_p^r}\underset{j\to\infty}{\to}0$. Then using the properties of the modulus

$$\omega_r(g,t)_p \le \omega_r(g-g_j,t)_p + \omega_r(g_j,t)_p$$
$$\le C\left(\left\|g-g_j\right\|_p + t^r\left|g_j\right|_{r,p}\right)$$
$$\underset{j\to\infty}{\to} Ct^r\left|g\right|_{r,p}.$$

□

[20%] Theorem: Let $f\in Lip(\alpha)$. Approximation with piecewise constants over uniform knots gives
$$E_N(f)_{L_\infty([0,1])} := \inf_{\phi\in S(N_1)^{1/N}}\left\|f-\phi\right\|_\infty \le CN^{-\alpha}\left|f\right|_{Lip(\alpha)}.$$

Proof Recall that for $g\in C^1[0,1]$, we constructed $\phi_g\in S(N_1)^{1/N}$, such that $E_N(g)_\infty \le \left\|g-\phi_g\right\|_\infty \le N^{-1}\left|g\right|_{1,\infty}$. Therefore, for any $g\in C^1[0,1]$
$$\left\|f-\phi_g\right\|_\infty \le \left\|f-g\right\|_\infty + \left\|g-\phi_g\right\|_\infty$$
$$\le \left\|f-g\right\|_\infty + N^{-1}\left|g\right|_{1,\infty}.$$
For a sequence $\{g_k\}$, with $K_1(f,N^{-1})_\infty = \lim_{k\to\infty}\left\{\left\|f-g_k\right\|_\infty + N^{-1}\left|g_k\right|_{1,\infty}\right\}$, we get

$$\left\|f-\phi_{g_k}\right\|_\infty \le \left\|f-g_k\right\|_\infty + N^{-1}\left|g_k\right|_{1,\infty}\underset{k\to\infty}{\to}K_1(f,N^{-1})_\infty.$$

Using the equivalence of the modulus of smoothness and K-functional,

$$E_N(f)_\infty \le K_1(f,N^{-1})_\infty$$
$$\le C\omega_1(f,N^{-1})_\infty$$
$$\le CN^{-\alpha}\left|f\right|_{Lip(\alpha)}.$$

□

Denote $E_N(f)_p := \inf_{P\in\Pi_N}\left\|f-P\right\|_{L_p(\mathbb{T})}$. Here, we shall assume we are approximating real functions. This implies we can use real trigonometric polynomials.

[40%] Theorem For a periodic function $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ and any $r \ge 1$,

$$E_N(f)_p \le C(r)\,\omega_r\left(f, N^{-1}\right)_p$$

Proof Recall the Fejér kernel of degree $m-1$

$$K_m(t) = \frac{1}{m}\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^2 .$$

We construct the approximating trigonometric polynomial using the **Jackson kernel**

$$J_{N,r}(t) := \lambda_{N,r}\left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2r}, \qquad \int_{\mathbb{T}} J_{N,r}(t)\,dt = 1, \qquad m := \left\lfloor \frac{N}{r} \right\rfloor + 1.$$

It is a positive, symmetric kernel, trigonometric polynomial of degree $\le N$ because

$$r(m-1) = r\left\lfloor \frac{N}{r} \right\rfloor \le r\frac{N}{r} = N .$$

Also, since it is an even trigonometric polynomial, we can write it as

$$J_{N,r}(x) = \sum_{k=0}^{N} a_k \cos(kx).$$

The actual approximating polynomial is of a more sophisticated form of convolution

$$S_{N,r}(f,x) := \int_{\mathbb{T}} \left[(-1)^{r+1}\Delta_t^r(f,x) + f(x)\right] J_{N,r}(t)\,dt .$$

Notice that $\int_{\mathbb{T}} \left(-f(x) + f(x)\right) J_{N,r}(t)\,dt = 0$. This means that $S_{N,r}(f,x)$ is a combination of terms

$$\int_{\mathbb{T}} f(x+kt)\cos(lt)\,dt, \qquad k = 1,\ldots,r, \qquad l = 0,\ldots,N .$$

We want to show that $S_{N,r}(f,x) \in \Pi_N(\mathbb{T})$. Now $f(x+kt)$ as a function of t has period $2\pi/k$. This means that $\int_{\mathbb{T}} f(x+kt)\cos(lt)\,dt = 0$, unless k divides l. To see this let $g(t)$ have period $2\pi/k$. Then for any $l \in \mathbb{Z}$

$$\int_0^{2\pi} g(t)e^{ilt}dt = \int_{2\pi/k}^{2\pi+2\pi/k} g(t)e^{ilt}dt = \int_0^{2\pi} g(y+2\pi/k)e^{il(y+2\pi/k)}dy = \int_0^{2\pi} g(y)e^{il(y+2\pi/k)}dy = e^{i\frac{2\pi l}{k}}\int_0^{2\pi} g(y)e^{ily}dy .$$

$$\Rightarrow e^{i\frac{2\pi l}{k}} = 1 \;\; or \;\; \int_0^{2\pi} g(y)e^{ily}dy = 0$$

Thus, we get for k that divides l

$$\int_{\mathbb{T}} f(x+kt)\cos(lt)\,dt = \frac{1}{k}\int_{x-\pi k}^{x+\pi k} f(y)\cos\left(\frac{l}{k}(y-x)\right)dy$$
$$= \int_0^{2\pi} f(y)\cos\left(\frac{l}{k}(y-x)\right)dy$$
$$= \left(\int_0^{2\pi} f(y)\cos\left(\frac{l}{k}y\right)dy\right)\cos\left(\frac{l}{k}x\right) + \left(\int_0^{2\pi} f(y)\sin\left(\frac{l}{k}y\right)dy\right)\sin\left(\frac{l}{k}x\right)$$

So $S_{N,r}(f,x)$ is composed of trigonometric polynomial terms of degree $\le N$. Now we use the following:

(i) $\omega_r(f,t)_p = \omega_r\left(f,\frac{Nt}{N}\right)_p \le (Nt+1)^r\,\omega_r\left(f,\frac{1}{N}\right)_p$

(ii) Lemma 7.2.1 in Constructive Approximation shows that $\int_0^{\pi} t^k J_{N,r}(t)\,dt \le C(r)N^{-k}$, $k = 0,\ldots,2r-2$.

Therefore

$$\left\|S_{N,r}(f,x) - f\right\|_p = \left\|\int_{\mathbb{T}}\left((-1)^{r+1}\Delta_t^r(f,\cdot) + f(\cdot) - f(\cdot)\right)J_{N,r}(t)\,dt\right\|_p$$
$$\le \left\|\int_{\mathbb{T}}\left|\Delta_t^r(f,\cdot)\right| J_{N,r}(t)\,dt\right\|_p$$
$$\underset{\text{Minkowski}}{\le} \int_{\mathbb{T}}\omega_r(f,|t|)_p\,J_{N,r}(t)\,dt$$
$$\le \omega_r\left(f,\frac{1}{N}\right)_p\int_{\mathbb{T}}(N|t|+1)^r J_{N,r}(t)\,dt$$
$$= 2\omega_r\left(f,\frac{1}{N}\right)_p\sum_{k=0}^{r}\binom{r}{k}N^k\int_0^{\pi} t^k J_{N,r}(t)\,dt$$
$$\le C(r)\,\omega_r\left(f,\frac{1}{N}\right)_p.$$

□

Continuous definition of Besov space Let $\alpha > 0$, $0 < q, p \le \infty$. Let $r \ge \lfloor \alpha \rfloor + 1$. The Besov space $B_q^\alpha(L_p(\Omega))$ is the collection of functions $f \in L_p(\Omega)$ for which

$$|f|_{B_q^\alpha(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{-\alpha}\omega_r(f,t)_p, & q = \infty. \end{cases}$$

is finite. The norm is

$$\|f\|_{B_q^\alpha(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + |f|_{B_q^\alpha(L_p(\Omega))}.$$

[20%] Theorem $W_p^m \subseteq B_q^\alpha(L_p)$, $\forall \alpha < m$, $1 \le p \le \infty$, $0 < q \le \infty$.

Proof Let $g \in W_p^m(\Omega)$. This implies $g \in L_p(\Omega)$. We have that $r := \lfloor \alpha \rfloor + 1 \le m$. It is sufficient to take the integral of the Besov-semi norm over $[0,1]$.

$$\int_0^1 \left[t^{-\alpha} \omega_r(g,t)_p \right]^q \frac{dt}{t} \le C \int_0^1 \left[t^{-\alpha} t^r |g|_{r,p} \right]^q \frac{dt}{t}$$
$$\le C |g|_{r,p}^q \int_0^1 t^{(r-\alpha)q-1} dt$$
$$\le C |g|_{r,p}^q .$$

Then

$$\|g\|_{B_q^\alpha(L_p(\Omega))} \le C \left(\|g\|_{L_p(\Omega)} + \left(\int_0^1 \left[t^{-\alpha} \omega_r(g,t)_p \right]^q \frac{dt}{t} \right)^{1/q} \right)$$
$$\le C \left(\|g\|_{L_p(\Omega)} + |g|_{r,p} \right)$$
$$\le C \|g\|_{W_p^r(\Omega)} .$$

□

[20%] Theorem One has the following equivalent form of the Besov semi-norm

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \begin{cases} \left(\sum_{k=-\infty}^{\infty} \left[2^{k\alpha} \omega_r(f, 2^{-k})_p \right]^q \right)^{1/q}, & 0 < q < \infty. \\ \sup_{k \in \mathbb{Z}} 2^{k\alpha} \omega_r(f, 2^{-k})_p , & q = \infty. \end{cases}$$

Proof Define $\varphi(t) := t^{-\alpha} \omega_r(f,t)_p$. Then we claim that for $t \in \left[2^{-k-1}, 2^{-k}\right]$, $k \in \mathbb{Z}$, we have

$$2^{-r} \varphi(2^{-k}) \le \varphi(t) \le 2^\alpha \varphi(2^{-k}).$$

To see that, we use the following properties:

(i) $\omega_r(f,t)_p$ is non-decreasing

(ii) $\omega_r(f,Nt)_p \le N^r \omega_r(f,t)_p$

The left-hand side

$$2^{-r} \varphi(2^{-k}) = 2^{k\alpha - r} \omega_r(f, 2^{-k})_p = 2^{k\alpha - r} \omega_r(f, 22^{-k-1})_p$$
$$\underset{\text{(ii)}}{\le} 2^{k\alpha - r} 2^r \omega_r(f, 2^{-k-1})_p \underset{\text{(i)}}{\le} 2^{k\alpha} \omega_r(f,t)_p \le t^{-\alpha} \omega_r(f,t)_p$$

The right-hand side

$$t^{-\alpha} \omega_r(f,t)_p \underset{\text{(i)}}{\le} t^{-\alpha} \omega_r(f, 2^{-k})_p \le 2^{(k+1)\alpha} \omega_r(f, 2^{-k})_p \le 2^\alpha \varphi(2^{-k})$$

This gives us for $0 < q < \infty$, $k \in \mathbb{Z}$

$$\int_{2^{-k-1}}^{2^{-k}} \varphi(t)^q \frac{dt}{t} \sim \varphi\left(2^{-k}\right)^q \int_{2^{-k-1}}^{2^{-k}} \frac{dt}{t} \sim \varphi\left(2^{-k}\right)^q \Rightarrow \int_{2^{-k-1}}^{2^{-k}} \left(t^{-\alpha}\omega_r\left(f,t\right)_p\right)^q \frac{dt}{t} \sim \left[2^{k\alpha}\omega_r\left(f,2^{-k}\right)_p\right]^q.$$

□

[30%] Theorem Let $f(x) = \mathbf{1}_{\tilde{\Omega}}(x)$, $\tilde{\Omega} \subset [0,1]^n$, a domain with smooth boundary. Then $f \in B_\tau^\alpha$, $\alpha < 1/\tau$.

Proof For $\Omega = [0,1]^n$, with $l(Q)$ denoting the level of the cube Q, we may take the sum over $k \geq 0$

$$|f|_{B_\tau^\alpha} \sim \left(\sum_{Q \in D, l(Q) \geq 0} \left(|Q|^{-\alpha/n}\omega_r\left(f,Q\right)_\tau\right)^\tau\right)^{1/\tau}.$$

For any Q, we have that $\omega_r\left(f,Q\right)_\tau = 0$, if $\partial\tilde{\Omega} \cap Q = \emptyset$. Otherwise, if $l(Q) = k$,

$$\omega_r\left(f,Q\right)_\tau \leq C\|f\|_{L_\tau(Q)} \leq C\left(\int_Q 1^\tau\right)^{1/\tau} = C|Q|^{1/\tau} = C2^{-kn/\tau}.$$

Therefore,

$$\begin{aligned} |f|_{B_\tau^\alpha}^\tau &\leq C\sum_{l(Q)\geq 0}\left(|Q|^{-\alpha/n}\omega_r\left(f,Q\right)_\tau\right)^\tau \\ &\leq C\sum_{k=0}^{\infty}\left(2^{k\alpha}2^{-kn/\tau}\right)^\tau \#\left\{Q : l(Q) = k,\ Q \cap \partial\tilde{\Omega} \neq \emptyset\right\} \\ &= C\sum_{k=0}^{\infty} 2^{k(\alpha\tau - n)} \#\left\{Q : l(Q) = k,\ Q \cap \partial\tilde{\Omega} \neq \emptyset\right\} \end{aligned}$$

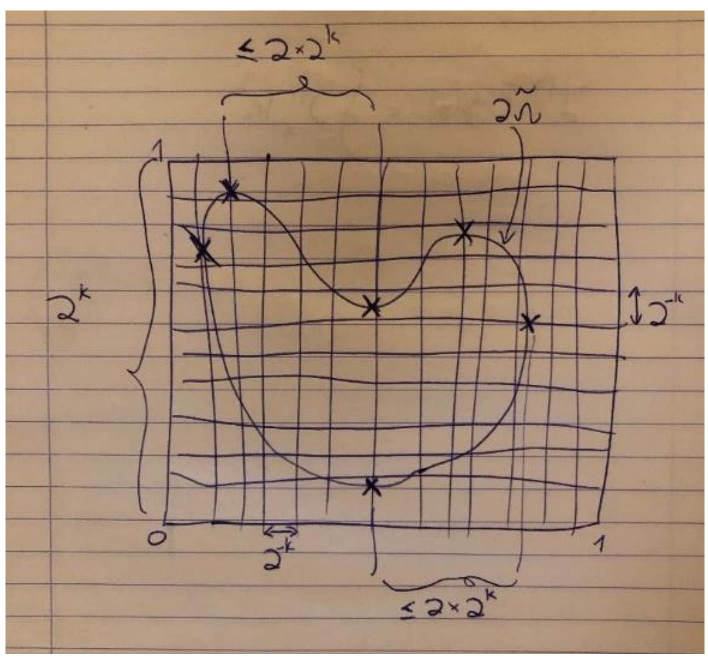
We argue that

$$\#\left\{Q : l(Q) = k,\ Q \cap \partial\tilde{\Omega} \neq \emptyset\right\} \leq c\left(\tilde{\Omega}\right)2^{k(n-1)}. \quad (*)$$

This implies that if $\alpha < 1/\tau$

$$|f|_{B_\tau^\alpha}^\tau \leq C\sum_{k=0}^{\infty} 2^{k(\alpha\tau-n)}2^{k(n-1)} = C\sum_{k=0}^{\infty} 2^{k(\alpha\tau-1)} < \infty.$$

Let's get back to the estimate (*). Let use show a picture argument for $\tilde{\Omega} \subset [0,1]^2$. There is a finite number of points where the gradient of the boundary of the domain is aligned with one of the main axes. Between these points, the boundary segments are monotone in x_1 and x_2, and therefore can only intersect at most 2×2^k dyadic cubes.



[30%] Theorem Suppose that for $r \ge 1$ and a bounded $\varphi : \mathbb{R} \to \mathbb{R}$, with sufficiently fast decay, there exist linear functionals $\{\tilde{g}_k\}$ on $\Pi_{r-1}(\mathbb{R})$, such that for any univariate polynomial $\tilde{P} \in \Pi_{r-1}(\mathbb{R})$,

$$\tilde{P}(x) = \sum_{k \in \mathbb{Z}} \tilde{g}_k(P)\varphi(x-k).$$

Then for $\phi(x) := \prod_{i=1}^{n} \varphi(x_i)$, there exist linear functionals $\{g_k\}$, such that for any $P \in \Pi_{r-1}(\mathbb{R}^n)$

$$P(x) = \sum_{k \in \mathbb{Z}^n} g_k(P)\phi(x-k).$$

Proof Let $x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i}$, with $\alpha_i \le r-1$. Since φ reproduces univariate polynomials, for $1 \le i \le n$,

$$x_i^{\alpha_i} = \sum_{k_i} \tilde{g}_{k_i}\left(x_i^{\alpha_i}\right)\varphi(x_i - k_i).$$

This gives

$$\begin{aligned}
x^\alpha &= \prod_{i=1}^{n}\left(\sum_{k_i} \tilde{g}_{k_i}\left(x_i^{\alpha_i}\right)\varphi(x_i-k_i)\right) \\
&= \sum_{k \in \mathbb{Z}^n} \prod_{i=1}^{n} \tilde{g}_{k_i}\left(x_i^{\alpha_i}\right)\varphi(x_i-k_i) \\
&= \sum_{k \in \mathbb{Z}^n} \underbrace{\left(\prod_{i=1}^{n} \tilde{g}_{k_i}\left(x_i^{\alpha_i}\right)\right)}_{=:g_k(x^\alpha)} \phi(x-k) \\
&= \sum_{k \in \mathbb{Z}^n} g_k\left(x^\alpha\right)\phi(x-k),
\end{aligned}$$

where we define

$$g_k\left(x^\alpha\right) := \left(\prod_{i=1}^{n} \tilde{g}_{k_i}\left(x_i^{\alpha_i}\right)\right).$$

Now, for any $P \in \Pi_{r-1}(\mathbb{R}^n)$, $P(x) = \sum_{|\alpha|<r} a_\alpha x^\alpha$, we define

$$g_k(P) := \sum_{|\alpha|<r} a_\alpha g_k\left(x^\alpha\right).$$

This gives

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^n} g_k(P)\phi(x-k) &= \sum_{k \in \mathbb{Z}^n}\left(\sum_{|\alpha|<r} a_\alpha g_k\left(x^\alpha\right)\right)\phi(x-k) \\
&= \sum_{|\alpha|<r} a_\alpha \sum_{k \in \mathbb{Z}^n} g_k\left(x^\alpha\right)\phi(x-k) \\
&= \sum_{|\alpha|<r} a_\alpha x^\alpha = P(x).
\end{aligned}$$

□

Define

$$K_h(x,y) := h^{-n} K\left(h^{-1}x, h^{-1}y\right), \ \ h>0,$$

and

$$T_h f(x) := \int_{\mathbb{R}^n} K_h(x,y) f(y)\,dy.$$

[30%] Theorem Assume a kernel operator $K(x,y)$ satisfies for $r \ge 1$

(i) $P(x) = \int_{\mathbb{R}^n} K(x,y) P(y)\,dy$, $\forall P \in \Pi_{r-1}\left(\mathbb{R}^n\right)$,

(ii) $\left|K(x,y)\right| \le c \dfrac{1}{\left(1+|x-y|\right)^{n+r+\varepsilon}}$, for some $\varepsilon > 0$ and any $x, y \in \mathbb{R}^n$,

Then, for $g \in C^r\left(\mathbb{R}^n\right)$

$$\left\|g - T_h g\right\|_\infty \le C h^r |g|_{r,\infty}, \quad h>0.$$

Proof Recall the Taylor polynomial $T_{r-1,x} g(y) := \sum_{|\alpha|<r} \dfrac{\partial^\alpha g(x)}{\alpha!}(y-x)^\alpha \in \Pi_{r-1}$,

The estimate of Taylor remainder $\left|R_{r,x} g(y)\right| \le c|y-x|^r \max_{z\in B(x,|y-x|)} \max_{|\alpha|=r} \left|\partial^\alpha g(z)\right|$.

For $p=\infty$

$$\begin{aligned}
\left|g(x) - \int_{\mathbb{R}^n} K(x,y) g(y)\,dy\right| &= \left|g(x) - \int_{\mathbb{R}^n} K(x,y)\left(T_{r-1,x} g(y) + R_{r,x} g(y)\right) dy\right| \\
&= \left|g(x) - \int_{\mathbb{R}^n} K(x,y) T_{r-1,x} g(y)\,dy - \int_{\mathbb{R}^n} K(x,y) R_{r,x} g(y)\,dy\right| \\
&= \left|g(x) - T_{r-1,x} g(x) - \int_{\mathbb{R}^n} K(x,y) R_{r,x} g(y)\,dy\right| \\
&= \left|g(x) - g(x) - \int_{\mathbb{R}^n} K(x,y) R_{r,x} g(y)\,dy\right| \\
&= \left|\int_{\mathbb{R}^n} K(x,y) R_{r,x} g(y)\,dy\right|
\end{aligned}$$

$$\begin{aligned}
\left|g(x) - \int_{\mathbb{R}^n} K(x,y) g(y)\,dy\right| &\le \int_{\mathbb{R}^n} \left|K(x,y)\right| \left|R_{r,x} g(y)\right| dy \\
&\le C|g|_{r,\infty} \int_{\mathbb{R}^n} \left|K(x,y)\right| |y-x|^r\,dy \\
&\le C|g|_{r,\infty} \int_{\mathbb{R}^n} \frac{1}{\left(1+|y-x|\right)^{n+r+\varepsilon}} |y-x|^r\,dy \\
&\le C|g|_{r,\infty} \int_{\mathbb{R}^n} \frac{1}{\left(1+|y-x|\right)^{n+\varepsilon}}\,dy \\
&\le C|g|_{r,\infty}.
\end{aligned}$$

Let $h>0$. Then for $\tilde{g}(x) = g(hx)$

$$\begin{aligned}
\left\|g(hx) - \int_{\mathbb{R}^n} K(x,y) g(hy)\,dy\right\|_\infty &= \left\|\tilde{g}(x) - \int_{\mathbb{R}^n} K(x,y)\tilde{g}(y)\,dy\right\|_\infty \\
&\le C|\tilde{g}|_{r,\infty} = C\left|g(h\cdot)\right|_{r,\infty} = Ch^r |g|_{r,\infty}
\end{aligned}$$

Therefore

$$\begin{aligned}\left\| g(x) - \int_{\mathbb{R}^n} K_h(x,y) g(y) dy \right\|_\infty &= \left\| g(x) - h^{-n} \int_{\mathbb{R}^n} K\left(h^{-1}x, h^{-1}y\right) g(y) dy \right\|_\infty \\ &= \left\| g(x) - \int_{\mathbb{R}^n} K\left(h^{-1}x, z\right) g(hz) dz \right\|_\infty \\ &= \left\| g(hx) - \int_{\mathbb{R}^n} K(x,z) g(hz) dz \right\|_\infty \\ &\le Ch^r \left| g \right|_{r,\infty}.\end{aligned}$$

□

The sinc is defined by $\phi(x) = \phi(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{\sin(\pi x_i)}{\pi x_i}$. We have

$$\hat{\phi}(w) = \hat{\phi}(w_1,\dots,w_n) = 1_{[-\pi,\pi]^n}(w) = \prod_{i=1}^{n} 1_{[-\pi,\pi]}(w_i).$$

[20%] Theorem The shifts of the sinc $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^n}$ are an ortho-basis for $S(\phi)$.

Proof Observe that $\{\phi(\cdot - k)\}_{k=-\infty}^{\infty}$ is an orthonormal basis of its span $\Leftrightarrow \langle \phi(\cdot - k), \phi(\cdot - j)\rangle = \delta_{k,j}$, $\forall k, j \in \mathbb{Z}^n$ $\Leftrightarrow$ $\langle \phi, \phi(\cdot + j)\rangle = \delta_{0,j}$, $\forall j \in \mathbb{Z}^n$. We now compute using Parseval

$$\begin{aligned}\langle \phi, \phi(\cdot + j)\rangle &= (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi}(w) \overline{\left(\phi(\cdot + j)\right)^{\wedge}(w)} dw \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\hat{\phi}(w)\right|^2 e^{-ijw} dw \\ &= (2\pi)^{-n} \int_{[-\pi,\pi]^n} e^{-ijw} dw \\ &= \delta_{0,j}.\end{aligned}$$

□

[30%] Theorem Let $P_{S(\phi)^h}$ be the orthogonal projector onto $S(\phi)^h$, where ϕ is the sinc function. Then, for $f \in L_2(\mathbb{R}^n)$

$$\left(P_{S(\phi)^h} f\right)^{\wedge} = \hat{f}(w) 1_{\left[-h^{-1}\pi, h^{-1}\pi\right]^n}(w), \quad h > 0.$$

Proof Since $\langle \phi(\cdot - k), \phi(\cdot - j)\rangle = \delta_{k,j}$, we have that $\phi_{h,k}(x) := h^{-n/2} \phi\left(h^{-1}x - k\right)$ satisfy $\langle \phi_{h,k}, \phi_{h,j}\rangle = \delta_{k,j}$. Thus, $\{\phi_{h,k}\}$ is an ortho-basis of $S(\phi)^h$.

$$\left(P_{S(\phi)^h} f\right)^{\wedge}(w)=\left(\sum_k \left\langle f,\phi_{h,k}\right\rangle \phi_{h,k}\right)^{\wedge}(w)$$
$$= h^{n/2}\hat{\phi}(hw)\sum_k \left\langle f,\phi_{h,k}\right\rangle e^{-ihkw}$$
$$= \mathbf{1}_{\left[-h^{-1}\pi,h^{-1}\pi\right]^n}(w)\sum_k \frac{h^{n/2}}{(2\pi)^n}\left\langle \hat{f},\hat{\phi}_{h,k}\right\rangle_{L^2(\mathbb{R}^n)} e^{-ihkw}$$
$$= \mathbf{1}_{\left[-h^{-1}\pi,h^{-1}\pi\right]^n}(w)\underbrace{\sum_k\left(\frac{1}{\left(2h^{-1}\pi\right)^n}\int_{\left[-h^{-1}\pi,h^{-1}\pi\right]^n}\hat{f}(z)e^{ihkz}dz\right)e^{-ihkw}}_{=\hat{f}(w),\quad w\in\left[-h^{-1}\pi,h^{-1}\pi\right]^n}$$

Why? Observe that $\left\{e^{ihkw}\right\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis of $L_2\left[-h^{-1}\pi,h^{-1}\pi\right]^n$ using the normalized dot-product

$$\langle h,g\rangle_h := \frac{1}{\left(2h^{-1}\pi\right)^n}\int_{\left[-h^{-1}\pi,h^{-1}\pi\right]^n} h(w)\overline{g(w)}dw .$$

Therefore, the restriction of $\hat{f}$ to $\left[-h^{-1}\pi,h^{-1}\pi\right]^n$ is represented by the Fourier series of $\left\{e^{ihkw}\right\}_{k\in\mathbb{Z}^n}$ and so, by the above computation

$$\left(P_{S(\phi)^h} f\right)^{\wedge} = \hat{f}(w)1_{\left[-h^{-1}\pi,h^{-1}\pi\right]^n}(w).$$

□

[40%] Theorem The sinc has 'infinite' / spectral approximation order, i.e., $\forall r\geq 1$, $\forall f\in W_2^r\left(\mathbb{R}^n\right)$,

$$E\left(f,S(\phi)^h\right)_2 \leq C(n,r)h^r\left|f\right|_{r,2}.$$

Proof First assume $f\in\mathcal{S}$. This means $f\in W_2^r\left(\mathbb{R}^n\right)$, $\forall r\geq 1$. We claim that

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2|w|^{2r}dw \leq C(n,r)\left|f\right|_{r,2}^2 = C(n,r)\left(\sum_{|\alpha|=r}\left\|D^\alpha f\right\|_2\right)^2$$

Let's start with $n=1$. In this case

$$\left(f^{(r)}\right)^{\wedge}(w)=(iw)^r\ \hat{f}(w)\Rightarrow\left|\left(f^{(r)}\right)^{\wedge}(w)\right|=|w|^r\left|\hat{f}(w)\right|.$$

So, by Parseval

$$\left|f\right|_{r,2}^2=\left\|f^{(r)}\right\|_2^2=\frac{1}{2\pi}\int_{\mathbb{R}}\left|\left(f^{(r)}\right)^{\wedge}(w)\right|^2dw=\frac{1}{2\pi}\int_{\mathbb{R}}\left|\hat{f}(w)\right|^2|w|^{2r}dw.$$

For $n\geq 2$, repeated application, coordinate by coordinate, each step similar to the univariate case, gives

$$\left(D^\alpha f\right)^{\wedge}(w)=(iw)^\alpha\ \hat{f}(w)\Rightarrow\left|\left(D^\alpha f\right)^{\wedge}(w)\right|=\left|w^\alpha\right|\left|\hat{f}(w)\right|.$$

This gives

$$\left\|D^\alpha f\right\|_2^2=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\left(D^\alpha f\right)^{\wedge}(w)\right|^2dw=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2\left|w^\alpha\right|^2dw.$$

Now, for $n \ge 2$

$$\left|w\right|^{2r} = \left(\sum_{m=1}^{n} w_m^2\right)^r = \left(w_1^r\right)^2 + n\left(w_1^{r-1}w_2\right)^2 + \cdots + \left(w_n^r\right)^2 = \sum_{|\alpha|=r} a_\alpha \left(w^\alpha\right)^2 .$$

Thus, we obtain

$$\begin{aligned}
\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2 \left|w\right|^{2r} dw &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2 \sum_{|\alpha|=r} a_\alpha \left|w^\alpha\right|^2 dw \\
&\le C(n,r)\sum_{|\alpha|=r}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(w)\right|^2 \left|w^\alpha\right|^2 dw \\
&= C(n,r)\sum_{|\alpha|=r}\left\|D^\alpha f\right\|_2^2 \\
&\underset{\|\cdot\|_{l_2}\le\|\cdot\|_{l_1}}{\le} C(n,r)\left(\sum_{|\alpha|=r}\left\|D^\alpha f\right\|_2\right)^2 \\
&= C(n,r)\left|f\right|_{r,2}^2 .
\end{aligned}$$

Now, for $h > 0$, $w \notin \left[-\pi/h, \pi/h\right]^n \Rightarrow h\left|w\right| \ge \pi$. Therefore

$$\begin{aligned}
E\left(f, S(\phi)^h\right)_2^2 = \left\|f - P_{S(\phi)^h} f\right\|_2^2 &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\setminus[-\pi/h,\pi/h]^n}\left|\hat{f}(w)\right|^2 dw \\
&\le \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\setminus[-\pi/h,\pi/h]^n}\left|hw\right|^{2r}\left|\hat{f}(w)\right|^2 dw \\
&\le h^{2r}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|w\right|^{2r}\left|\hat{f}(w)\right|^2 dw \\
&\le C(n,r)h^{2r}\left|f\right|_{r,2}^2 .
\end{aligned}$$

The general case of $f \in W_2^r\left(\mathbb{R}^n\right)$ is obtained by a density argument $\left\{f_k\right\}_{k=1}^\infty$, $f_k \in \mathcal{S}$, $\left\|f - f_k\right\|_{W_2^r} \underset{k\to\infty}{\to} 0$.

□

[20%] Theorem For $r \ge 1$, the univariate B-spline satisfies the two-scale relation

$$N_r(x) = \sum_{k=0}^{r} 2^{1-r}\binom{r}{k} N_r(2x-k).$$

Proof Recall that $N_r = \underbrace{N_1 * \cdots * N_1}_{r \text{ times}}$ and therefore

$$\hat{N}_r(w) = \left(\frac{1-e^{-iw}}{iw}\right)^r .$$

Assume that there exist $\left\{p_{r,k}\right\}_{k=0}^r$, such that

$$N_r(x) = \sum_{k=0}^{r} p_{r,k} N_r(2x-k).$$

This implies that

$$
\begin{aligned}
&\hat{N}_r(w) = \sum_{k=0}^{r} p_{r,k} \left(N_r(2\cdot - k)\right)^{\wedge}(w) \\
&\Leftrightarrow \left(\frac{1-e^{-iw}}{iw}\right)^r = \frac{1}{2}\left(\frac{1-e^{-i(w/2)}}{i(w/2)}\right)^r \sum_{k=0}^{r} p_{r,k} e^{-ikw/2} \\
&\Leftrightarrow \left(\left(1+e^{-i(w/2)}\right)\left(1-e^{-i(w/2)}\right)\right)^r = 2^{r-1}\left(1-e^{-i(w/2)}\right)^r \sum_{k=0}^{r} p_{r,k} e^{-ikw/2} \\
&\Leftrightarrow 2^{1-r}\left(1+e^{-i(w/2)}\right)^r = \sum_{k=0}^{r} p_{r,k} e^{-ikw/2} \\
&\Leftrightarrow p_{r,k} = 2^{1-r}\binom{r}{k}, \quad 0 \le k \le r.
\end{aligned}
$$

□

[30%] Theorem For any real $T_N \in \Pi_N(\mathbb{T})$,

$$T_N'(x)^2 + N^2 T_N(x)^2 \le N^2 \|T_N\|_\infty^2, \qquad x \in \mathbb{T}.$$

Corollary $\|T_N'\|_\infty \le N\|T_N\|_\infty$, and by repeated applications $\left\|T_N^{(r)}\right\|_\infty \le N^r \|T_N\|_\infty$.

Proof of theorem First assume $\|T_N\| < 1$. W.l.g we can assume $x = 0$, and that $T_N'(0) \ge 0$. Let α, $|\alpha| < \pi/(2N)$ such that $\sin N\alpha = T_N(0)$ and define,

$$S_N(y) := \sin N(y+\alpha) - T_N(y) \in \Pi_N.$$

At the points

$$y_k := -\alpha + \frac{(2k-1)\pi}{2N}, \qquad k = 0, \pm1, \pm2, \ldots, \pm N,$$

$$\operatorname{sign}(S_N(y_k)) = \operatorname{sign}\left(\underbrace{\sin\frac{(2k-1)\pi}{2}}_{=(-1)^{k+1}} - \underbrace{T_N(y_k)}_{|T_N(y_k)|<1}\right) = (-1)^{k+1}.$$

This means S_N has $2N$ zeros, with a unique zero at each interval (y_k, y_{k+1}). Next,

$$\left.\begin{aligned} (y_0, y_1) &= \left(-\alpha - \frac{\pi}{2N}, -\alpha + \frac{\pi}{2N}\right) \\ |\alpha| &< \frac{\pi}{2N} \end{aligned}\right\} \Rightarrow 0 \in (y_0, y_1)$$

Also $S_N(0) := \sin N\alpha - T_N(0) = 0$, $S_N(y_1) > 0$. If $S_N'(0) < 0$, then there must be another zero in $(0, y_1)$, which is a contradiction. Hence $S_N'(0) \ge 0$ and

$$0 \le T_N'(0) = N\cos N\alpha - S_N'(0)$$
$$\le N\cos N\alpha = N\sqrt{1-\sin^2 N\alpha} = N\sqrt{1-T_N(0)^2}.$$

This gives

$$T_N'(0)^2 \le N^2\left(1-T_N(0)^2\right) \Rightarrow T_N'(0)^2 + N^2 T_N(0)^2 \le N^2.$$

Now take arbitrary $T_N \ne 0$ and $\lambda > \|T_N\|_\infty$, and apply this relation to T_N/λ. Then

$$\frac{T_N'(0)^2}{\lambda^2} + N^2\frac{T_N(0)^2}{\lambda^2} \le N^2 \Rightarrow T_N'(0)^2 + N^2 T_N(0)^2 \le N^2\lambda^2$$
$$\underset{\lambda \to \|T_N\|}{\Rightarrow} T_N'(0)^2 + N^2 T_N(0)^2 \le N^2\|T_N\|^2.$$

□

Bernstein for non-uniform piecewise polynomials. Let

$$\Sigma_{N,r} := \left\{ \sum_{j=0}^{N-1} P_j \mathbf{1}_{[t_j, t_{j+1})} : T = \{t_j\},\ 0 = t_0 < t_1 < \cdots < t_N = 1,\quad P_j \in \Pi_{r-1} \right\}.$$

[20%] Lemma For any algebraic polynomial $P \in \Pi_{r-1}(\mathbb{R}^n)$, bounded convex $\Omega \subset \mathbb{R}^n$, and $0 < p, q \le \infty$,

$$\|P\|_{L_q(\Omega)} \sim |\Omega|^{1/q-1/p}\|P\|_{L_p(\Omega)},$$

with constants of equivalency that depend on p, q, n, r but not on the polynomial or domain.

Proof By John's Lemma, there exists an affine transformation, $Ax = Mx + b$, such that

$$B(0,1) \subseteq A^{-1}(\Omega) \subseteq B(0,n).$$

Observe that

$$A(B(0,1)) \subseteq \Omega \subseteq A(B(0,n)) \Rightarrow |\Omega| \sim |\det M|.$$

By the equivalency of same finite dimensional (quasi) Banach spaces, there exist constants depending only on p, q, n, r, such that for any $\tilde{P} \in \Pi_{r-1}(\mathbb{R}^n)$, $\|\tilde{P}\|_{L_p(B(0,1))} \sim \|\tilde{P}\|_{L_q(B(0,n))}$. Now, for $P \in \Pi_{r-1}$, denote $\tilde{P} := P(A\cdot) \in \Pi_{r-1}$. Then,

$$\|P\|_{L_q(\Omega)} = |\det M|^{1/q}\|\tilde{P}\|_{L_q(A^{-1}(\Omega))}$$
$$\le |\det M|^{1/q}\|\tilde{P}\|_{L_q(B(0,n))}$$
$$\le C|\det M|^{1/q}\|\tilde{P}\|_{L_p(B(0,1))}$$
$$\le C|\det M|^{1/q}\|\tilde{P}\|_{L_p(A^{-1}(\Omega))}$$
$$\le C|\Omega|^{1/q-1/p}\|P\|_{L_p(\Omega)}.$$

□

[30%] Theorem For $\varphi \in \Sigma_{N,r}$, $\frac{1}{\tau} = \alpha + \frac{1}{p}$, $0 < \alpha < r$,

$$|\varphi|_{B_\tau^\alpha} \le CN^\alpha \|\varphi\|_p .$$

Proof Let $P_j \in \Pi_{r-1}$ and $t > 0$. If $t < (t_{j+1} - t_j)/r$, we have seen we can estimate

$$\omega_r\left(P_j \mathbf{1}_{[t_j,t_{j+1})}, t\right)_\tau \le C\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_\infty t^{1/\tau}.$$

For $t \ge (t_{j+1} - t_j)/r$, we can bound by

$$\begin{aligned}\omega_r\left(P_j \mathbf{1}_{[t_j,t_{j+1})}, t\right)_\tau &\le C\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_\tau \\ &= C\left\|P_j\right\|_{L_\tau[t_j,t_{j+1}]} \\ &\le C\left\|P_j\right\|_{L_\infty[t_j,t_{j+1}]}\left(t_{j+1} - t_j\right)^{1/\tau}.\end{aligned}$$

Therefore, for $0 < \tau \le 1$ (the case $1 < \tau < \infty$ is similar)

$$\begin{aligned}\omega_r(\varphi,t)_\tau^\tau &\le \sum_{j=0}^{N-1} \omega_r\left(P_j \mathbf{1}_{[t_j,t_{j+1})}, t\right)_\tau^\tau \\ &\le C\sum_{j=0}^{N-1}\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_\infty^\tau \min\left(t, \left(t_{j+1} - t_j\right)/r\right).\end{aligned}$$

We apply the lemma for $q = \infty$,

$$\begin{aligned}|\varphi|_{B_\tau^\alpha}^\tau &= \int_0^\infty \left(t^{-\alpha}\omega_r(\varphi,t)_\tau\right)^\tau \frac{dt}{t} \\ &\le C\sum_{j=0}^{N-1}\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_\infty^\tau \int_0^\infty t^{-\alpha\tau}\min\left(t,\left(t_{j+1}-t_j\right)/r\right)\frac{dt}{t} \\ &\le C\sum_{j=0}^{N-1}\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_p^\tau \left(t_{j+1}-t_j\right)^{-\tau/p}\left(\int_0^{(t_{j+1}-t_j)/r} t^{-\alpha\tau}dt + \left(t_{j+1}-t_j\right)\int_{(t_{j+1}-t_j)/r}^\infty t^{-\alpha\tau-1}dt\right) \\ &\le C\sum_{j=0}^{N-1}\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_p^\tau \left(t_{j+1}-t_j\right)^{-\tau/p}\left(t_{j+1}-t_j\right)^{1-\alpha\tau} \\ &\underset{1=\alpha\tau+\tau/p}{=} C\sum_{j=0}^{N-1}\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_p^\tau \\ &\underset{Holder\ p>\tau}{\le} C\left(\sum_{j=0}^{N-1}\left(\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_p^\tau\right)^{p/\tau}\right)^{\tau/p} N^{1-\tau/p} \\ &= C\left(\sum_{j=0}^{N-1}\left\|P_j \mathbf{1}_{[t_j,t_{j+1})}\right\|_p^p\right)^{\tau/p} N^{1-\tau/p} \\ &= C\|\varphi\|_p^\tau N^{1-\tau/p}.\end{aligned}$$

Therefore

$$|\varphi|_{B_\tau^\alpha} \le C\|\varphi\|_p N^{1/\tau-1/p} = CN^\alpha\|\varphi\|_p .$$

□

Let $\Phi := \{\Phi_N\}_{N\ge 0}$, each Φ_N is a set of functions in a (quasi) Banach space X, satisfying:

(i) $0 \in \Phi_N$, $\Phi_0 := \{0\}$,

(ii) $\Phi_N \subset \Phi_{N+1}$,

(iii) $a\Phi_N = \Phi_N$, $\forall a \neq 0$,

(iv) $\Phi_N + \Phi_N \subset \Phi_{cN}$, for some constant $c(\Phi)$,

(v) $\overline{\bigcup_N \Phi_N} = X$,

(vi) Each $f \in X$ has a near best approximation from Φ_N. That is, there exists a constant $C(\Phi)$, such that for any N, one has $\varphi_N \in \Phi_N$,

$$\|f - \varphi_N\|_X \le CE_N(f)_X, \qquad E_N(f)_X := \inf_{\varphi \in \Phi_N} \|f - \varphi\|_X .$$

[30%] Theorem Let Y, X, $r > 0$, and $\Phi := \{\Phi_N\}$ as above. For the K-functional

$$K(f,t) := K(X,Y,f,t) := \inf_{g \in Y} \left\{ \|f - g\|_X + t|g|_Y \right\},$$

(i) If the Jackson inequality is satisfied

$$E_N(g)_X \le CN^{-r}|g|_Y, \qquad \forall g \in Y,$$

then

$$E_N(f)_X \le CK(f, N^{-r}), \quad f \in X, \ N = 1, 2, \ldots .$$

(ii) If the Bernstein inequality is satisfied

$$|\varphi|_Y \le CN^r \|\varphi\|_X, \qquad \forall \varphi \in \Phi_N,$$

then

$$K(f, 2^{-mr}) \le C2^{-mr} \sum_{k=0}^{m} 2^{kr} E_{2^k}(f).$$

Proof

(i) Let $f \in X$. By the Jackson theorem, for any $g \in Y$

$$E_N(f)_X \le \|f - g\|_X + E_N(g)_X \le C\left(\|f - g\|_X + N^{-r}|g|_Y\right).$$

We then take infimum over $g \in Y$.

(ii) Let $\varphi_k \in \Phi_{2^k}$, such that $\|f - \varphi_k\| \le CE_{2^k}(f)$, $k = 0,1,2\ldots$. Denote $\psi_0 = \varphi_0 = 0$, $\psi_k := \varphi_k - \varphi_{k-1}$, $k \ge 1$. Observe that by properties (iii), (iv), $\psi_k \in \Phi_{c2^k}$. Using the fact that $\{\varphi_k\}$ are near-best approximants

$$\|\psi_k\| \le \|f - \varphi_k\| + \|f - \varphi_{k-1}\| \le 2CE_{2^{k-1}}(f), \ k \ge 1.$$

Since $\varphi_m = \sum_{k=0}^{m} \psi_k$, $|\psi_0|_Y = 0$, it follows that

$$
\begin{aligned}
K\left(f,2^{-mr}\right) &\le \left\|f-\varphi_m\right\|_X + 2^{-mr}\left|\varphi_m\right|_Y \\
&\le C\left(E_{2^m}\left(f\right)+2^{-mr}\sum_{k=1}^{m}\left|\psi_k\right|_Y\right) \\
&\le C\left(E_{2^m}\left(f\right)+2^{-mr}\sum_{k=1}^{m}2^{kr}\left\|\psi_k\right\|_X\right) \\
&\le C\left(E_{2^m}\left(f\right)+2^{-mr}\sum_{k=1}^{m}2^{kr}E_{2^{k-1}}\left(f\right)\right) \\
&\le C2^{-mr}\sum_{k=0}^{m}2^{kr}E_{2^k}\left(f\right).
\end{aligned}
$$

□

[20%] Theorem $\left(L_p,W_p^r\right)_{\theta,q}=B_q^{\alpha}\left(L_p\right)$, $\alpha=\theta r$, $0<\theta<1$, $1\le p\le\infty$.

Proof

$$
\begin{aligned}
\int_0^1\left[t^{-\theta}K\left(f,t\right)\right]^q\frac{dt}{t} &= \int_0^1\left[t^{-\theta}K_r\left(f,t\right)_p\right]^q\frac{dt}{t} \\
&\sim \int_0^1\left[t^{-\alpha/r}\omega_r\left(f,t^{1/r}\right)_p\right]^q\frac{dt}{t} \qquad s=t^{1/r}\Rightarrow ds=\frac{1}{r}t^{1/r-1}dt\Rightarrow s^{-1}ds=\frac{1}{r}t^{-1}dt \\
&\underset{s=t^{1/r}}{\sim} \int_0^1\left[s^{-\alpha}\omega_r\left(f,s\right)_p\right]^q\frac{ds}{s} \\
&= \left|f\right|_{B_q^{\alpha}\left(L_p\right)}^q .
\end{aligned}
$$

□

[30%] Theorem If the Jackson and Bernstein inequalities are valid for $X,Y\left(r\right),\Phi$, then, for $0<\theta<1$, $\alpha:=\theta r$ $0<q\le\infty$, then

$$
A_q^{\alpha}\left(X\right)=A_q^{\theta r}\left(X\right)\sim\left(X,Y\right)_{\theta,q} .
$$

Proof Assume $f\in\left(X,Y\right)_{\theta,q}$. By a previous theorem, the Jackson inequality gives $E_{2^m}\left(f\right)_X\le CK\left(f,2^{-mr}\right)$. Therefore, with $\alpha:=\theta r$,

$$
\begin{aligned}
\left|f\right|_{A_q^{\alpha}}^q &\le C\sum_{m=0}^{\infty}\left[2^{m\alpha}E_{2^m}\left(f\right)\right]^q \\
&\le C\sum_{m=0}^{\infty}\left[2^{mr\theta}K\left(f,2^{-mr}\right)\right]^q \\
&\le C\left|f\right|_{\theta,q}^q .
\end{aligned}
$$

Now assume $f\in A_q^{\alpha}\left(X\right)$. We shall use the discrete Hardy inequality. In our case, we have $a_m=E_{2^m}\left(f\right)$, $b_m=K\left(f,2^{-mr}\right)$, for $m\ge0$, $a_m,b_m=0$, for $m<0$. Using the Bernstein inequality, we proved that with $\lambda:=r>\alpha=\theta r$

$$
K\left(f,2^{-mr}\right)\le C2^{-mr}\sum_{k=0}^{m}2^{kr}E_{2^k}\left(f\right).
$$

Therefore,

$$
\begin{aligned}
|f|_{\theta,q} &\le C\left(\sum_{m=0}^{\infty}\left[2^{mr\theta}K\left(f,2^{-mr}\right)\right]^{q}\right)^{1/q} \\
&\le C\left(\sum_{m=0}^{\infty}\left[2^{m\alpha}E_{2^m}\left(f\right)\right]^{q}\right)^{1/q} \\
&\le C|f|_{A_q^\alpha}.
\end{aligned}
$$

□