Introduction to approximation theory: Assignment II

1. Prove that if $S(\Phi)$ provides approximation order r for $1 \le p \le \infty$, then for any $f \in L_p(\mathbb{R}^n)$,

$$E\left(f, S(\Phi)^h\right)_p \le C\omega_r(f,h)_p, \qquad h > 0.$$

Hint: You may use the appropriate equivalence of the modulus and the K-functional.

2. Let $K(x,y) := \sum_{k\in\mathbb{Z}^n} \tilde{\phi}(y-k)\phi(x-k)$, $K_h(x,y) := h^{-n}K\left(h^{-1}x, h^{-1}y\right)$, for $h>0$, and $T_h f(x) := \int_{\mathbb{R}^n} K_h(x,y) f(y)dy$. Prove that $T_h f(x) \in S(\phi)^h$, $h>0$.

3. Prove that
   a. If there exist coefficients $\{p_k\}$ such that $\phi(x) = \sum_k p_k \phi(2x-k)$, then $S(\phi) \subset S(\phi)^{1/2}$.
   b. If $S(\phi) \subset S(\phi)^{1/2}$, then $S(\phi)^{2^{-j}} \subset S(\phi)^{2^{-(j+1)}}$, $\forall j \in \mathbb{Z}$.

4. Assume ϕ^*, ψ^* generate a univariate orthonormal multiresolution and a wavelet system, respectively. Prove that for $n=2$, the following is an orthonormal basis of $L_2(\mathbb{R}^2)$:

$$\left\{\psi^e_{j,k}\right\}, \qquad \psi^e_{j,k}(x) := 2^j \psi^e\left(2^j x - k\right), \qquad e = 1,2,3, \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^2,$$

   where

$$\psi^1(x_1,x_2) := \phi^*(x_1)\psi^*(x_2), \quad \psi^2(x_1,x_2) := \psi^*(x_1)\phi^*(x_2), \quad \psi^3(x_1,x_2) := \psi^*(x_1)\psi^*(x_2).$$