Foundations of approximation theory: Assignment II

1. Prove that if $S(\Phi)$ provides approximation order r for $1 \le p \le \infty$, then for any $f \in L_p(\mathbb{R}^n)$,

$$E\left(f, S(\Phi)^h\right)_p \le C\omega_r(f,h)_p, \qquad h>0.$$

2. Prove that if $S(\phi)$ is a PSI space, then $S(\phi)^{2^{-j}}$, $j \ge 0$ is a FSI space of length 2^{nj}. That is, as an integer shift invariant space, it is generated by at least 2^{nj} generators.

3. Let $K(x,y) := \sum_{k\in\mathbb{Z}^n} \tilde{\phi}(y-k)\phi(x-k)$, $K_h(x,y) := h^{-n}K\left(h^{-1}x, h^{-1}y\right)$, for $h>0$, and $T_h f(x) := \int_{\mathbb{R}^n} K_h(x,y) f(y)\,dy$. Prove that $T_h f(x) \in S(\phi)^h$, $h>0$.

4. Let $P_{S(\phi)^h}$ be the orthogonal projector onto $S(\phi)^h$, where ϕ is the sinc function. Prove that

$$\left(P_{S(\phi)^h} f\right)^{\wedge} = \hat{f}(w)1_{\left[-h^{-1}\pi, h^{-1}\pi\right]^n}(w), \quad h>0.$$

Hint (the case $h=1$) Let $g:\mathbb{T}^n \to \mathbb{C}$, $g(w) := \hat{f}(w)$, $w \in \mathbb{T}^n$,

$$\begin{aligned}
\left(P_{S(\phi)} f\right)^{\wedge}(w) &= \sum_{k\in\mathbb{Z}^n} \frac{1}{(2\pi)^n}\left\langle \hat{f}, \left(\phi(\cdot-k)\right)^{\wedge}\right\rangle \left(\phi(\cdot-k)\right)^{\wedge}(w) \\
&= \sum_{k\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n} \hat{f}(w)e^{ikw}dw\right)e^{-iwk}1_{[-\pi,\pi]^n}(w) \\
&= 1_{[-\pi,\pi]^n}(w)\sum_{k\in\mathbb{Z}^n}\hat{g}(-k)e^{-iwk}.
\end{aligned}$$

For general $h>0$, you will need to use the correct ortho-basis of $L_2\left(\left[-h^{-1}\pi, h^{-1}\pi\right]^n\right)$.

5. Prove that
   a. If there exist coefficients $\{p_k\}$ such that $\phi(x) = \sum_k p_k \phi(2x-k)$, then $S(\phi) \subset S(\phi)^{1/2}$.
   b. If $S(\phi) \subset S(\phi)^{1/2}$, then $S(\phi)^{2^{-j}} \subset S(\phi)^{2^{-(j+1)}}$, $\forall j \in \mathbb{Z}$.